Homework 9  
Mathematical foundations of informatics (I201, 2008)  
Instructor: Tang

(This HW will be collected on 12/3 Wed. in the class. Write LEGIBLY and explain your answers clearly. The homework you hand in must be your own work, IN YOUR OWN WORDS and your own explanation. NO late homework will be accepted. Problem 5 is optional.)

1. (16 pts) Consider a first order language that consists of one binary predicate symbol $R$. Let $\Psi$ be the formula: $\exists x \exists y \forall z((x \neq y) \land (R(x,z) \leftrightarrow R(y,z)))$.
   a. Let $U=\{a, b, c\}$ and $I(R)=\{(c,b),(b,b),(a,b)\}$, is $\Psi$ valid in this model $M=\{U, I\}$? Justify your answer.
   b. Let $U'=\{a, b, c\}$ and $I'(R)=\{(c,b),(b,b),(b,a)\}$, is $\Psi$ valid in this model $M=\{U', I'\}$? Justify your answer.

2. (18 pts) Consider a first order language that consists of one binary predicate $T$ and one unary predicate $W$. Assume the model $M = \{U, I\}$, where $U = \{a, b, c\}$, $I(T) = \{(a, a), (a, b), (b, b)\}$, $I(W) = \{b, c\}$. Tell whether each of the following formulas is valid under the model $M$.
   a. $\forall x(\exists y T(x,y) \land W(x))$
   b. $\forall x(W(x) \rightarrow \exists y T(x,y))$
   c. $\exists x \exists y ((x \neq y) \land W(x) \land \neg T(x, y))$

3. (16 pts) Consider a first order language that consists of one binary predicate symbol $R$ and one unary predicate symbol $T$. Assuming $U=\{a, b, c, d\}$, for each of the following formulas, find an interpretation $I$ to make it true, and another interpretation $I'$ to make it false. Be precise in your reasoning and explain your answer. Note: you need to define both $I(T)$ and $I(R)$ for $I$.
   a. $\exists x(T(x) \land \forall y R(x,y))$
   b. $\forall x \exists y (T(x) \rightarrow \neg R(x,y))$

4. (10 extra pts) Show the arguments below are universally valid by using set theory.
   a. $\exists x P(x) \rightarrow \forall x Q(x)$
      
      $P(a)$
      \[ \frac{Q(a)}{\text{P(a)}} \]

   b. $\forall x (\neg P(x) \lor Q(x))$
      
      $P(a)$
      \[ \frac{Q(a)}{\text{P(a)}} \]