2. onto (surjective).

You should convince yourself that if these two properties hold, then it is always going to be the case that f^{-1} is a function. In order to do this, you should remember the definition of a function, and of one-to-one and onto.

14.6 Counting the Number of Relations

Given sets A and B, how many relations are there from A to B? Since a relation R from A to B is simply a subset of $A \times B$, there are as many relations from A to B, as there are different subsets of $A \times B$. So now we've modified the questions to how many subsets of $A \times B$ are there? We know that $\mathcal{P}(A \times B)$ is the set that contains all of the subsets of $A \times B$. Therefore, we just need to know how big the set $\mathcal{P}(A \times B)$ is. Looking back in our notes, we know that $|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{|A| \cdot |B|}$, and therefore the size of the set is $2^{|A| \cdot |B|}$.

15 Graph Theory

In this section we will introduce graphs. As we will see, graphs have a lot of similarities with certain types of relations, and are another great way to represent data. Graphs have the nice property that they are easy to visualize, and yet general enough that they allow us to represent a lot of real world relations between objects, and other relations of interest.

15.0.1 What is a Graph?

There are two types of graphs: **directed and undirected**. They have similar, but different definitions.

An **undirected graph** G consists of a pair (V, E), where V is a set of *vertices or nodes*, and $E \subseteq \{\{a, b\} | a, b \in V\}$ is a set of *undirected edges.*

Below we give several examples of undirected graphs.

1. $G = (V, E)$ where $V = \{\alpha, \beta, \gamma\}$ and $E = \{\{\alpha, \beta\}, \{\beta, \gamma\}\}$.
2. $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{\{1\}, \{2, 1\}, \{2, 3\}, \{4, 3\}, \{4\}, \{4, 1\}\}$

We now consider Some basic definition related to undirected graphs. Given an undirected graph $G = (V, E)$:

1. Two vertices $a, b \in V$ are **neighbors** if there exits an edge $e = \{a, b\} \in E$.
2. For every edge $e = \{a, b\} \in E$ we say that the vertices a and b are **incident** to the edge e.
3. For any $a \in V$ we call an edge $e = \{a, a\} = \{a\} \in E$ a **loop**.
4. We call an undirected graph withouts loops a **simple undirected graph**. (Note that example 1 above is simple, whereas example 2 is not)

A **directed graph** G consists of a pair (V, E) where V is a set of vertices or nodes, and $E \subseteq V \times V$ is a set of *directed edges.*

Below we give several examples of directed graphs.

1. $G = (V, E)$ where $V = \{\alpha, \beta, \gamma\}$ and $E = \{(\alpha, \beta), (\beta, \gamma), (\alpha, \alpha), (\beta, \alpha)\}$.

2. $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 1), (3, 3), (4, 3), (4, 1)\}$

Similar to the case of an undirected graph, we define the notions on neighbors, incidence and loops. The notions have a natural correspondence to the corresponding notions in the case of undirected graphs. Given an undirected graph $G = (V, E)$:

1. We say that two vertices $a, b \in V$ are **neighbors** if there exits an edge $e = (a, b) \in E$.
2. For every edge $e = (a, b) \in E$ we say that the vertices a and b are **incident** to the edge e.
3. For any $a \in V$ we call an edge $e = (a, a) \in E$ a **loop**.
4. We call a directed graph without loops a **simple directed graph** . (Note that example 1 above is not simple, whereas example 2 is).

15.1 How do I Draw a Graph?

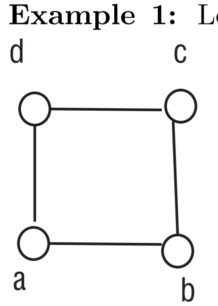
15.1.1 Undirected Graphs

Given an undirected graph $G = (V, E)$ we draw the graph as follows:

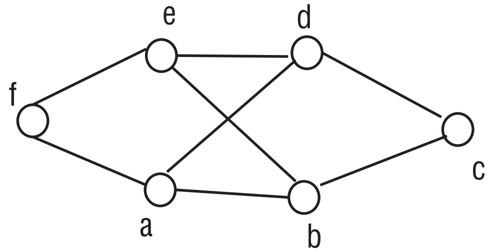
1. For each vertex $v \in V$ we draw a small circle to represent the vertex and label it v.
2. For each edge $\{a, b\} \in E$ where $a \neq b$ we draw a line from that begins at vertex a and ends at vertex b
3. For each edge $\{a\} = \{a, a\} \in E$ we draw a loop that starts and ends at vertex a.

Below are some examples, which we will refer to throughout the remainder of these notes:

- **Example 1:** Let $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{b, c\}, \{d, c\}, \{a, d\}$



- **Example 2:** Let $G = (V, E)$ where $V = \{a, b, c, d, e, f\}$ and $E = \{\{a, b\}, \{b, c\}, \{d, c\}, \{d, e\}, \{e, f\}, \{f, a\}, \{a, d\}, \{e, b\}\}$



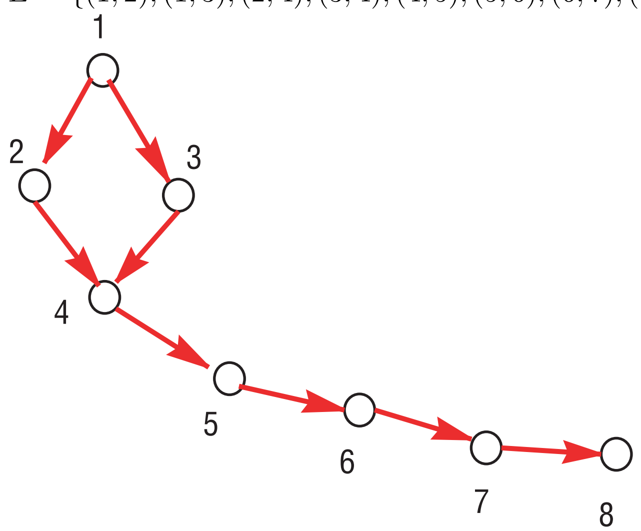
15.1.2 Directed Graphs

Given a directed graph $G = (V, E)$ we draw the graph as follows:

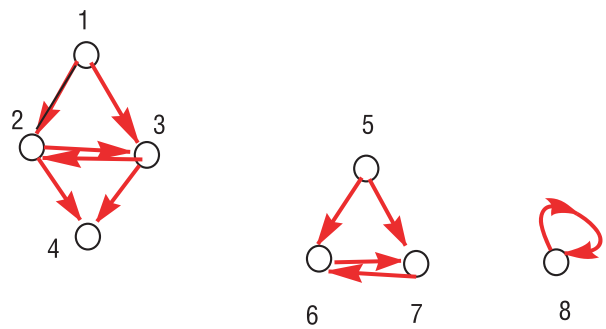
1. For each vertex $v \in V$ we draw a small circle to represent the vertex and label it v.
2. For each edge $(a, b) \in E$ where $a \neq b$ we draw an arrow that begins at vertex a and ends pointing at vertex b
3. For each edge $(a, a) \in E$ we draw an arrowed loop that starts and ends at vertex a.

Below are some more examples, which we will refer to throughout the remainder of these notes:

- **Example 3:** Let $G = (V, E)$ where $V = \{x | x \in \mathbb{N} \wedge 1 \leq x \leq 8\}$ and $E = \{(1, 2), (1, 3), (2, 4), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8)\}$



- **Example 4:** Let $G = (V, E)$ where $V = \{x | x \in \mathbb{N} \wedge 1 \leq x \leq 8\}$ and $E = \{(1, 2), (1, 3), (2, 4), (3, 4), (2, 3), (3, 2), (5, 6), (5, 7), (6, 7), (7, 6), (8, 8)\}$



15.2 Definitions of Graph Properties

15.2.1 Degree of a vertex

Given an **undirected graph** $G = (V, E)$ we define the degree of a vertex v as the sum of the number of non-loop edges incident to v plus two if there happens to exist a loop on the vertex v. Informally, this means we add the number of edges attached to the vertex, and count a loop as two edges since it is connected to the edge twice. More formally, we define it as:

$$\deg(v) = |\{\{v, a\}|\{v, a\} \in E \wedge b \neq v\}| + 2\,|\{\{v\}|\{v\} \in E\}|\,.$$

Note that if the graph is simple, then we can simple define it as:

$$\deg(v) = |\{\{v, a\}|\{v, a\} \in E \wedge b \neq v\}|$$

Examples:

- Each vertex in Example 1 has degree 2.
- In Example 2, vertices a, b, d and e have degree 3, and vertices c and f have degree 2.

Given a **undirected graph** $G = (V, E)$ we define three notions of degree of a vertex v: in-degree, out-degree and total degree. These are defined below:

in-degree: The in-degree of a vertex v is, informally, the number of arrows that enter (point towards) the vertex v. This is formally denoted as $in - deg(v) = |\{(u, v)|(u, v) \in E\}|$.

Examples:

- In Example 3, $in - deg(1) = 0, in - deg(2) = 1, in - deg(3) = 1, in - deg(4) = 2, in - deg(5) = 1$, and we leave the remainder of the vertices as an exercise.
- In Example 4, $in - deg(1) = 0, in - deg(2) = 2, in - deg(3) = 2, in - deg(4) = 2, in - deg(8) = 1$, and we leave the remainder of the vertices as an exercise.

out-degree: The out-degree of a vertex v is, informally, the number of arrows that leave (point away from) the vertex v. This is formally denoted as $out - deg(v) = |\{(v,u)|(v,u) \in E\}|$.

Examples:

- In Example 3, $out - deg(1) = 2, out - deg(2) = 1, out - deg(3) = 1, out - deg(4) = 1, out - deg(8) = 0$, and we leave the remainder of the vertices as an exercise.
- In Example 4, $out - deg(1) = 2, out - deg(2) = 2, out - deg(3) = 2, out - deg(4) = 0, out - deg(8) = 1$, and we leave the remainder of the vertices as an exercise.

15.2.2 Paths & Circuits

- A **path of length ℓ on an undirected graph** $G = (V, E)$ is a sequence of vertices on a graph $v_0, v_1, v_2, ..., v_\ell$ where for every i where $0 \leq i < \ell$ it's the case that $\{v_i, v_{i+1}\} \in E$. We can say this is a path from v_0 to v_ℓ or, vice-versa, from v_ℓ to v_0, as the graph is undirected so you could clearly travel the path in either direction.

  Informally, this means you could start at vertex v_0 and then follow an edge from it to v_2, and follow another edges from v_2 to v_3, etc....

  Examples:

  - In Example 1, a, b, c, d is an example of a path of length 3.
  - In Example 2, e, b, c, d, a, b is an example of a path of length 5.
  - In example 1, a is an example of a path of length 0.

- A **path of length ℓ on an directed graph** $G = (V, E)$ is a sequence of vertices on a graph $v_0, v_1, v_2, ..., v_\ell$ where for every i where $0 \leq i < \ell$ it's the case that $(v_i, v_{i+1}) \in E$. This path is said to go from v_0 to v_ℓ, as the direction is clearly important in this case, and you cannot necessarily travel the path in both directions.

  Informally, this means you could start at vertex v_1 and then an outgoing arrow to v_2, and follow another outgoing arrow from v_2 to v_3, etc....

  Examples:

  - In Example 3, $1, 2, 4, 5, 6, 7, 8$ is an example of a path of length 6.
  - In Example 3, $1, 2, 4, 3, 1$ is not a path, as there is no directed edge going from 4 to 3, not from 3 to 1.
  - In Example 4, $5, 6, 7, 6, 7, 6, 7$ is an example of a path of length 6.
  - In Example 4, $8, 8$ is an examples of a path of length 1, in contrast to 8 which is a path of length 0.

On both a directed graph and an undirected a **path of length ℓ is simple** if given the sequence $v_0, v_1, ..., v_\ell$ it's the case that there are no numbers i and j, where $i \neq j$ and $0 \leq i, j < \ell$ such that $v_i = (v_j$. This means that if you never cross the same vertex twice in your path.

On both a directed graph and an undirected graph a **a circuit of size ℓ** is given by a path $v_0, v_1, ..., v_\ell$ with the property that $v_0 = v_\ell$.

Examples:

- In Example 1, a, b, c, d is an example of a simple path.
- In Example 4, $5, 6, 7, 6, 7, 6, 7$ is an example of path that is not simple, since the edges $(7, 6)$ and $(6, 7)$ are both crossed multiple times.

15.2.3 Distance

Given a directed or undirected graph $G = (V, E)$, we say that the **distance** between two vertices a and b, is the length of the shortest path from a to b on G. We denote the distance between a and b as $dist(a, b)$. Therefore, on an **undirected graph for any vertices a and b it holds that $dist(a, b) = dist(b, a)$, but in the case of a directed graph it is not necessarily the case.**

If there is no path between a and b on G then we say that the distance between a and b is infinite, and write it as $dist(a, b) = \infty$.

Examples:

- In Example 1, $dist(a, d) = 1$, and $dist(a, c) = 2$.
- In Example 2, $dist(f, c) = 3$, and $dist(e, b) = 1$.
- In Example 3, $dist(1, 4) = 2$, $dist(4, 1) = \infty$, $dist(3, 4) = \infty$.
- In Example 4, $dist(8, 8) = 0$ (Note, there is a path $8, 8$ which would traverse the loop, but this path has length 1, and so is not as short as the path 8.

15.2.4 Diameter of a Graph

Given a directed or undirected graph $G = (V, E)$, we say that its **diameter** is the length of the longest shortest path between any two vertices in V. Thought of another way, suppose you always take the shortest paths to travel between any two points on the graph G, then what is the longest path you would ever have to take to get from the two vertices on the graph that are the furthest apart. Formally, the diameter of the graph is denoted by $diam(G)$, and

$$diam(G) = \max_{a,b \in V} \{x | x = dist(a, b)\}.$$

Examples:

- In Example 1, $diam(G) = 2$, as $dist(a, c) = 2$ and $dist(b, d)$.
- In Example 2, $diam(G) = 3$ as $dist(f, c) = 3$
- In Example 3, $diam(G) = \infty$ as $dist(4, 1) = \infty$.
- **If we added an edge $(8, 1)$ to the graph of Example 3** then the $diam(G) = 6$ as $dist(4, 2) = 6$.

15.2.5 Connected Graphs

- We say an **undirected graph $G = (V, E)$ is connected** if its diameter is finite.
- We say a **directed graph $G = (V, E)$ is connected** if for every pair of vertices a and b in V it is the case that either $dist(a, b)$ is finite or $dist(b, a)$ is finite.

Examples:

- The graphs in Examples 1,2 and 3 are connected.
- The graph in Example 4 is not connected.